INVERSES OF FUZZY RELATIONS. APPLICATION TO POSSIBILITY DISTRIBUTIONS AND MEDICAL DIAGNOSIS*

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In this paper we define and explore the properties of lower and upper inverses of fuzzy relations which extend multi-valued mappings. With the notion of degree of inclusion of non-fuzzy sets, we then relate the preceding notions to possibility distributions in natural languages and to problems of medical diagnosis.

Key Words: Multi-valued fuzzy relations, Lower inverses, Upper inverses, Ponderated inverses, Inverse possibility distributions, Medical diagnosis assistance.

1. Introduction

In approximate reasoning, fuzzy relations have shown to be of a primordial importance. Human judgments are often based on comparisons between couples of faced data. We study here a special class of fuzzy relations which extend multi-valued mappings, so that their membership (or compatibility) functions deal with subsets of given sets, rather than with single elements.

For any fuzzy relation, a lower and an upper inverses are defined. By means of a SUP–MIN composition they are associated with relations of inclusion and intersection for non fuzzy sets.

For practical purposes, the definition of a fuzzy relation of inclusion allows us to ponderate the upper inverse of a fuzzy relation. Such ponderated inverse contains the lower inverse and it is included in the upper inverse.

Reasoning now in natural languages, from inverses of fuzzy relations we derive inverse possibility distributions, which are induced by propositions in the form “X is F”, where F is a fuzzy relation and X a variable that takes values in the cartesian product of power sets of universes of discourse.

Finally it is shown how inverses of fuzzy relations may apply to a “medical knowledge” with degrees of associations between diagnosis and syndromes.

2. Fuzzy relations. Lower and upper inverses

Let two sets X and Y be given. Any non-fuzzy relation R between the elements of X and the elements of Y may acquire a functional nature, in the

sense of a multi-valued mapping from \( X \) to \( Y \), by specifying the elements \( y \) in \( Y \)
which are related to a given element \( x \) in \( X \), i.e. such that \((x, y)\) belongs to \( R \).

This simply means that the relation is being regarded as a correspondence from
\( X \) to \( Y \) and our purpose is now to study fuzzy correspondences between
non-fuzzy sets.

**Definition 1.** A *multi-valued* fuzzy relation \( R \) from \( X \) to \( Y \) is characterized by its
membership (or compatibility) function

\[
\mu_R : X \times 2^Y \rightarrow [0, 1]
\]

where \( 2^Y \) stands for the power set of \( Y \).

\( \mu_R \) is not a multi-valued mapping, but the fuzzy relation \( R \) is said “multi-valued” for its membership function associates a point in \( X \) and a subset of \( Y \)
with a degree of compatibility in the interval \([0, 1]\).

\( \mu_R \) is an extension of fuzzy or non-fuzzy relations from \( X \) to \( Y \), but the main
point of view here is that \( \mu_R \) generalizes multi-valued mappings from \( X \) to \( Y \).

**Definition 2.** For each subset \( A \) of \( X \), we define

\[
\forall B \subseteq Y, \quad \mu_R(A, B) = \sup_{x \in A} \mu_R(x, B).
\]

Note that, by convention, \( \sup_{x \in A} \mu_R(x, B) = 0 \) if \( A = \emptyset \).

**Definition 3.** Let \( R \) be a “multi-valued” fuzzy relation from \( X \) to \( Y \) and let \( A \) be
a fuzzy subset of \( X \), we define \( R \circ A \), a fuzzy subset of \( 2^Y \), by

\[
\forall B \subseteq Y, \quad \mu_{R \circ A}(B) = \sup_{x \in X} [\mu_A(x) \wedge \mu_R(x, B)].
\]

Definition 3 still holds when \( A \) is a non-fuzzy subset of \( X \) and in such a case, from Definition 2 we derive

\[
\mu_R(A, B) = \mu_{R \circ A}(B),
\]

and

\[
\forall x \in X, \forall B \subseteq Y, \quad \mu_R([x], B) = \mu_R(x, B),
\]

so that we shall now only consider fuzzy relations from \( 2^X \) to \( 2^Y \). When writing
that \( R \) is a fuzzy relation from \( X \) to \( Y \) it will be understood that its compatibility
function \( \mu_R \) is defined on \( 2^X \times 2^Y \) and that it is \([0, 1]\) valued. The notation
\( R \subseteq 2^X \times 2^Y \) will be used for it.

**Definition 4.** Let \( R \) be a fuzzy relation from \( X \) to \( Y \). We define its *lower inverse*

as the fuzzy relation \( R_* \) from \( Y \) to \( X \) (i.e. \( R_* \subseteq 2^Y \times 2^X \)) characterized by

\[
\forall B \subseteq Y, \forall A \subseteq X, \quad \mu_{R_*}(B, A) = \sup_{C \subseteq Y, C \neq B, C \subseteq B} \mu_R(A, C).
\]

(1)
**Definition 5.** Let \( R \) be a fuzzy relation from \( X \) to \( Y \). We define its *upper inverse* as the fuzzy relation \( R^* \) from \( Y \) to \( X \) (i.e. \( R^* \subseteq 2^Y \times 2^X \)) characterized by

\[
\forall B \subseteq Y, \forall A \subseteq X, \quad \mu_{R^*}(B, A) = \sup_{C \subseteq Y, C \cap B \neq \emptyset} \mu_R(A, C).
\]

(2)

For \( R \) and \( S \) fuzzy relations from \( X \) to \( Y \), defining

\[
R \subseteq S \quad \text{iff} \quad \forall A \subseteq X, \forall B \subseteq Y, \quad \mu_R(A, B) \leq \mu_S(A, B),
\]

one then has

\[
R \subseteq S \Rightarrow R^* \subseteq S^*.
\]

(3)

\[
R \subseteq S \Rightarrow R^* \subseteq S^*.
\]

(4)

With \( B_1 \) and \( B_2 \) being subsets of \( Y \), for any fuzzy relation \( R \) from \( X \) to \( Y \), one has

\[
B_1 \subseteq B_2 \Rightarrow \forall A \subseteq X, \quad \mu_{R^*}(B_1, A) \leq \mu_{R^*}(B_2, A),
\]

(5)

\[
B_1 \subseteq B_2 \Rightarrow \forall A \subseteq X, \quad \mu_R(B_1, A) \leq \mu_R(B_2, A).
\]

(6)

The following property holds for any fuzzy relation \( R \) from \( X \) to \( Y \).

\[
\forall A \subseteq X, \forall B \subseteq Y, \quad \mu_R(A, B) \leq \mu_{R^*}(B, A) \leq \mu_{R^*}(B, A),
\]

(7)

where the inequality in the right is equivalent to

\[
R^* \subseteq R^*.
\]

(8)

**Remarks.** Definitions 4 and 5 are an extension of the definitions of lower and upper inverses of multi-valued mappings.

The reason why we have introduced two kinds of inverses, a lower and an upper one, is because the fuzzy relations we study in this paper are considered as extensions of multi-valued mappings.

In Definition 4, the supremum is taken over the non void subsets \( C \) of \( Y \) which are included in \( B \) but, depending on applications, one may choose an order relation different from the one of inclusion in \( 2^Y \).

**Example.** Let \( X = Y \) be the set of possible positions in the game of chess. A position consists of the coordinates of the different pieces on the chess-board and the player whose move is next.

The set \( X \) is then the union of three disjoint sets \( X_1, X_2, X_0; \) \( X_1 \) is the set of positions in which White can move, \( X_2 \) is the set of positions in which Black can move and \( X_0 \) is the set of positions of checkmate or stalemate, when it is not possible for either to move.

\(^1\)This example is derived from an example in [2], where the author considers multi-valued mappings.
If $x \in X_1$ (resp. $X_2$) we shall denote by $\mu_R(x, B)$ the degree of combination of $x$ with the set of positions $B$, $B \subseteq X$, such that points in $B$ are immediately reachable by White (resp. Black) after position $x$; this determines a fuzzy relation from $X - X_1$ to $X$.

Starting with the position $x$, $\mu_{R^*}(B, \{x\}) = \sup_{C \subseteq Y, C \neq \emptyset, C \subseteq B} \mu_R(x, C)$ denotes the higher degree of association of $x$ with sets of positions which can only be in $B$ in the following move, while $\mu_{R^*}(B, \{x\}) = \sup_{C \subseteq Y, C \cap B \neq \emptyset} \mu_R(x, C)$ denotes the higher degree of association of $x$ with sets $C$ of positions that will have a common part with $B$ in the following move. To take into account the role that such common parts may play in relation with the extent of $C$ and $B$, it already seems natural to introduce weights to ponderate the expression of the upper inverse (see Section 5 for it).

3. First and second monotonocity. Composition of fuzzy relations

**Definition 6.** Let $R$ be a fuzzy relation from $X$ to $Y$. We say that $R$ has the property of first monotonocity or $R$ is 1-monotonous if, and only if, for $A_1$ and $A_2$ subsets of $X$,

$$A_1 \subseteq A_2 \Rightarrow \forall B \subseteq Y, \, \mu_R(A_1, B) \leq \mu_R(A_2, B).$$

(9)

**Definition 7.** Let $R$ be a fuzzy relation from $X$ to $Y$. We say that $R$ has the property of second monotonocity or $R$ is 2-monotonous if, and only if, for $B_1$ and $B_2$ subsets of $Y$,

$$B_1 \subseteq B_2 \Rightarrow \forall A \subseteq X, \, \mu_R(A, B_1) \leq \mu_R(A, B_2).$$

(10)

Let $R$ be a fuzzy relation from $X$ to $Y$, the following properties hold:

$R_*$ and $R^*$ are 1-monotonous.

(11)

If $R$ is 2-monotonous, then

$$\forall B \subseteq Y, \forall A \subseteq X, \, \mu_{R^*}(B, A) = \mu_R(A, B)$$

(12)

If $R$ is 2-monotonous, then

$$\forall B \subseteq Y, \forall A \subseteq X, \, \mu_{R^*}(B, A) = \mu_R(A, Y).$$

(13)

**Definition 8.** Let $Q$ be a fuzzy relation from $X$ to $Y$ and let $R$ be a fuzzy relation from $Y$ to $Z$ we define $R \circ Q$, a fuzzy relation from $X$ to $Z$, by

$$\forall A \subseteq X, \forall C \subseteq Z, \, \mu_{R \circ Q}(A, C) = \sup_{B \subseteq Y} [\mu_Q(A, B) \wedge \mu_R(B, C)].$$

(14)
Proposition. Let \( Q \) be a fuzzy relation from \( X \) to \( Y \) and let \( R \) be a fuzzy relation from \( Y \) to \( Z \). If \( Q \) and \( R \) are 2-monotonous, then
\[
(R \circ Q)_* = Q_* \circ R_*.
\]

Let us now define two non fuzzy relations \( S_\# \) and \( S^\# \), from \( Y \) to \( Y \) by \( \forall C \subseteq Y, \forall B \subseteq Y, \)
\[
\mu_{S_\#}(C, B) = \begin{cases} 1 & \text{for } C \subseteq B, C \neq \emptyset, \\ 0 & \text{elsewhere,} \end{cases}
\]
\[
\mu_{S^\#}(C, B) = \begin{cases} 1 & \text{for } C \cap B \neq \emptyset, \\ 0 & \text{elsewhere.} \end{cases}
\]

Let \( R \) be a fuzzy relation from \( X \) to \( Y \), then
\[
\forall B \subseteq Y, \forall A \subseteq X, \quad \mu_{R_*}(B, A) = \mu_{S_\# \circ R}(A, B).
\]
\[
\forall B \subseteq Y, \forall A \subseteq X, \quad \mu_{R^*}(B, A) = \mu_{S^\# \circ R}(A, B).
\]
\[
S_\#, S^\#, S_\# \circ R \text{ and } S^\# \circ R \text{ are 2-monotonous.}
\]
\[
R_* = (S_\# \circ R)_* \quad \text{(19)}
\]
\[
R^* = (S^\# \circ R)_* \quad \text{(20)}
\]
\[
\text{If } R \text{ is 2-monotonous, then } S_\# \circ R = R. \quad \text{(22)}
\]

4. Fuzzy relation of inclusion

Let us now define a fuzzy relation \( S \), from \( Y \) to \( Y \), which characterizes the degree of inclusion of couples of subsets of \( Y \). It is introduced for applications, mainly for the ponderation of the upper inverse of a fuzzy relation,

\[
\forall (\text{non-fuzzy}) \text{ subsets } C \text{ and } B \text{ of } Y, \quad \mu_{S}(C, B) \triangleq \text{ degree to which } C \subseteq B. \quad \text{(23)}
\]

Such a relation \( S \) will be called a fuzzy relation of inclusion in \( Y \).

We require \( S \) to verify the following four conditions

(i) \( \mu_S(C, B) = 0 \) for \( C \cap B = \emptyset, \)
(ii) \( \mu_S(C, B) = 1 \) for \( C \subseteq B, \ C \neq \emptyset, \)
(iii) \( 0 < \mu_S(C, B) < 1 \) (with the understanding that \( \mu_S(C, B) \) expresses the degree to which \( C \subseteq B \)) for \( C \cap B \neq \emptyset \) and \( C \) not included in \( B, \)
(iv) If \( B_1 \subseteq B_2 \), then \( \mu_S(C, B_1) \leq \mu_S(C, B_2) \), which is a natural condition, according to (23). This condition means that \( S \) is 2-monotonous.

\(^2\) The symbol \( \triangleq \) stands for "denotes" or is "equal by definition".
For example, if $Y$ is a finite set and if $n(D)$ denotes the cardinality of a (non fuzzy) subset $D$ of $Y$, the fuzzy relation $S$ from $Y$ to $Y$ defined by for all subsets $C$ and $B$ of $Y$,

$$
\mu_S(C, B) = \begin{cases} 
\frac{n(C \cap B)}{n(C)} & \text{for } C \neq \emptyset, \\
0 & \text{for } C = \emptyset,
\end{cases}
$$

verifies the four conditions in (24).

Remarks. The relation $S_#$, defined in (15), is a non fuzzy case of $S$. With $S_#$ we only deal with inclusion or not, but with $S$ we allow a possible continuity of variation for the degrees of inclusion.

Let $y$ be an element of $Y$ and let $B$ be a subset of $Y$. $\mu_S(\{y\}, B)$ is degree to which $\{y\} \subseteq B$, i.e. degree to which $y \in B$.

$$\mu_S(\{y\}, B) = \chi_B(y) = \begin{cases} 
1 & \text{for } y \in B \\
0 & \text{for } y \notin B,
\end{cases}
$$

where $\chi_B$ is the characteristic function of $B$, $\chi_B : Y \rightarrow \{0, 1\}$. In our example, see (25), $\{y\} \neq \emptyset$, so that

$$\mu_S(\{y\}, B) = \frac{n(\{y\} \cap B)}{n(\{y\})} = n(\{y\} \cap B) = \begin{cases} 
1 & \text{for } y \in B, \\
0 & \text{for } y \notin B.
\end{cases}
$$

For some applications, it may be useful to introduce the following type of relation of inclusion: $T$, fuzzy relation to type "x", with compatibility function $\mu_T : 2^Y \times [0, 1]^Y \rightarrow [0, 1]$, where $[0, 1]^Y$ defines the fuzzy subsets of $Y$, and such that for all (non-fuzzy) subset $C$ of $Y$ and for all fuzzy subset $B$ of $Y$, $\mu_T(C, B)$ is degree to which $C \subseteq B$.

For $y \in Y$ and for $B \in [0, 1]^Y$, $\mu_T(\{y\}, B)$ can be assimilated with the grade of membership of $y$ in $B$, say $\mu_B(y)$ in the usual notation.

Let us remark that in some problems, it may be more convenient to describe fuzziness in terms of a relation of inclusion, rather than in terms of a relation of membership.

Let us now note some properties related to $S$.

$$S_# \subseteq S \subseteq S^#.$$  \hspace{1cm} (26)

$$\forall B \subseteq Y, \forall C \subseteq Y, \quad \mu_{S^#}(B, C) = \mu_S(C, B).$$  \hspace{1cm} (27)

For all fuzzy relation $Q$ from $X$ to $Y$, $S \circ Q$ is a 2-monotonous fuzzy relation from $X$ to $Y$ and $\mu_{(S \circ Q)_#}(C, A) = \mu_{S \circ Q}(A, C)$ for all subsets $A$ of $X$ and $C$ of $Y$. 

5. Ponderation of the upper inverse of a fuzzy relation

For each subset $B$ of $Y$, let us define

$$Y_1(B) = \{ C \subseteq Y \mid C \subseteq B, C \neq \emptyset \},$$

$$Y_2(B) = \{ C \subseteq Y \mid C \cap B \neq \emptyset \text{ and } C \subseteq B \},$$

$$Y_3(B) = \{ C \subseteq Y \mid C \cap B = \emptyset \}. $$

From (24) we derive, for $B \subseteq Y$,

$$\mu_{\mu}(C, B) = 1 \quad \text{for} \quad C \in Y_1(B),$$

$$\mu_{\mu}(C, B) \in ]0, 1[ \quad \text{for} \quad C \in Y_2(B),$$

$$\mu_{\mu}(C, B) = 0 \quad \text{for} \quad C \in Y_3(B).$$

Moreover,

$$2^Y = Y_1(B) \cup Y_2(B) \cup Y_3(B), \quad \text{for all} \quad B \subseteq Y. \quad (35)$$

From Definitions 4 and 5 we now derive, for all $B \subseteq Y$ and for all $A \subseteq X$,

$$\mu_{R*}(B, A) = \sup_{C \in Y_1(B)} \mu_R(A, C), \quad (36)$$

$$\mu_{R**}(B, A) = \mu_{R*}(B, A) \vee \left[ \sup_{C \in Y_2(B)} \mu_R(A, C) \right]. \quad (37)$$

In the right-hand member of (37), the supremum is taken over the subsets $C$ of $Y$ which have a common part with $B$ and which are not included in $B$. The need to take into account the way such $C$'s intersect with $B$ lead us to introduce the following ponderation.

**Definition 9.** Let $R$ be a fuzzy relation from $X$ to $Y$. We define its **ponderated upper inverse** as the fuzzy relation $R^{**}$ from $Y$ to $X$ (i.e. $R^{**} \subseteq 2^Y \times 2^X$) characterized by, for all $B \subseteq Y$ and for all $A \subseteq X$,

$$\mu_{R^{**}}(B, A) = \mu_{R*}(B, A) \vee \left[ \sup_{C \in Y_2(B)} (\mu_R(A, C) \land \mu_S(C, B)) \right], \quad (38)$$

where $Y_2(B)$ is defined in (30) and $S$ is a fuzzy relation of inclusion verifying the four conditions in (24).

It follows easily that

$$R_* \subseteq R^{**} \subseteq R^*.$$  

**Proposition.** If $R$ be a fuzzy relation from $X$ to $Y$, then

$$R^{**} = (S \circ R)_*.$$  

(40)
One may note that (40) is equivalent to
\[ \forall B \subseteq Y, \forall A \subseteq X, \quad \mu_{R \circ}(B, A) = \mu_{S \circ}(A, B), \] (41)
which has to be considered in parallel with the similar expressions given in (17) and (18).

6. Fuzzy relations and possibility distributions

The theory of possibility related to fuzzy sets was introduced by Zadeh in [13].

Let \( X \) be a variable which takes values in a universe of discourse \( U \) with "\( X = u \)" signifying that \( X \) is assigned the value \( u, u \in U \). Let \( F \) be a fuzzy subset of \( U \), characterized by a membership function \( \mu_f \), then the proposition "\( X \) is \( F \)" induces a possibility distribution, \( \Pi_X \), which is postulated to be equal to \( F \).

Correspondingly, the possibility distribution function associated with \( X \) (or the possibility distribution function of \( \Pi_X \)) is denoted by \( \pi_X \) and is defined to be equal to \( \mu_F \).

\[
\text{Poss}(X = u \mid X \text{ is } F) = \mu_F(u) = \pi_X(u) \tag{42}
\]
with the understanding that \( \Pi_X \) is induced by "\( X \) is \( F \)". For example,

\[
\text{Poss} \left( \text{Height (John)} = 175 \mid \text{John is tall} \right) = \mu_{\text{tall}}(175),
\]
where the universe of discourse \( U \) represents heights in cm, "Height" is an implied attribute of \( X \) (i.e. John), and "tall" is a fuzzy subset of \( U \), characterized by a given membership function \( \mu_{\text{tall}} \).

Let us now consider propositions involving \( n \) variables \( X_1, \ldots, X_n \), with \( X_i \) taking values in \( U_i, i = 1, \ldots, n \), which are \( n \) possibly different universes of discourse. Let \( F \) be a usual fuzzy relation in the cartesian product \( U = U_1 \times \cdots \times U_n \) and let \( X = (X_1, \ldots, X_n) \). Then the proposition "\( X \) is \( F \)" translates into \( \Pi_X = F \)

\[
X \text{ is } F \rightarrow \Pi_{(X_1, \ldots, X_n)} = F, \tag{43}
\]
where \( \Pi_{(X_1, \ldots, X_n)} \) is an \( n \)-ary possibility distribution which is induced by the proposition "\( X \) is \( F \)". Correspondingly, the possibility distribution function associated with \( X \) is given by

\[
\forall (u_1, \ldots, u_n) \in U, \quad \pi_{(X_1, \ldots, X_n)}(u_1, \ldots, u_n) = \mu_F(u_1, \ldots, u_n), \tag{44}
\]
where \( \mu_F \) is the membership function of \( F \).

\[
\text{Poss}(X \mid X \text{ is } F) = \mu_{\Pi_X} = \pi_X = \mu_F. \tag{45}
\]

Let us now relate possibility distributions to inverses of 2-ary fuzzy relations. Let \( X = (X_1, X_2) \) and let \( R \) be a usual fuzzy relation from \( U_1 \) to \( U_2 \),

\[
X \text{ is } R \rightarrow \Pi_{(X_1, X_2)} = R. \tag{46}
\]
Our purpose is to define \((H_\chi)^{-1}\), an inverse possibility distribution of \(H_\chi\).

Let \(X^{-1} \triangleq (X_2, X_1)\) and let us consider a proposition of the form "\(X^{-1}\) is \(R^{-1}\)", where \(R^{-1}\) is the inverse\(^3\) of \(R\), a usual fuzzy relation from \(U_2\) to \(U_1\) which is characterized by

\[
\forall (u_2, u_1) \in U_2 \times U_1, \quad \mu_{R^{-1}}(u_2, u_1) = \mu_R(u_1, u_2).
\]

The inverse possibility distribution of \(H_\chi\), associated with \(X^{-1}\), is postulated to be equal to \(H_\chi^{-1}\). In other words, from \(\Pi_{(X_1, X_2)} = R\) one may deduce

\[
\Pi_{(X_2, X_1)} = R^{-1} \quad \text{or} \quad \Pi_{X^{-1}} = (\Pi_\chi)^{-1}
\]

where \(R^{-1}\) is characterized in (47).

Let us now suppose that \(U_1\) and \(U_2\) are two power sets of universes of discourses, i.e. \(U_1 = 2^{V_1}\) and \(U_2 = 2^{V_2}\), and that \(R\) is a fuzzy relation from \(V_1\) to \(V_2\), in the sense given in the preceding sections \((R \subseteq 2^{V_1} \times 2^{V_2})\).

Starting from \(\Pi_{(X_1, X_2)} = R\), \(\Pi_{(X_2, X_1)}\) is no more defined uniquely, like in (48).

From \(\Pi_{(X_1, X_2)} = R\) we now postulate the existence of two inverse possibility distributions of \(\Pi_\chi\), which are associated with \(X^{-1}\), a lower and an upper one

\[
\Pi_{\chi_{X_2,X_1}} = R_{\leq},
\]

\[
\Pi_{\chi_{X_2,X_1}} = R_{\geq},
\]

where \(R_{\leq} \subseteq U_2 \times U_1\) and \(R_{\geq} \subseteq U_2 \times U_1\) (recall that \(U_1 = 2^{V_1}\) and \(U_2 = 2^{V_2}\)) are characterized by

\[
\mu_{R_{\leq}}(u_2, u_1) = \sup_{w \in U_2} \mu_R(u_1, w)
\]

\[
\mu_{R_{\geq}}(u_2, u_1) = \sup_{w \in U_1, w \cap U_2 \neq \emptyset} \mu_R(u_1, w).
\]

7. Fuzzy relations and medical diagnosis

The following application is based on fuzzy relations in the sense given in the preceding sections. We assume here that the source of imprecision is not in the nature of symptoms and diagnosis, but rather in the correspondences which associate syndromes with groups of diagnosis.

In a given pathology, let \(\mathcal{S}\) be a set of symptoms and let \(\mathcal{D}\) be a set of diagnosis, all being well-defined. We shall denote by \(s\) (resp. \(d\) or \(d'\)) generic subsets of \(\mathcal{S}\) (resp. \(\mathcal{D}\)).

\(^3\) We recall that such inverse relations are of a great use in the resolution of SUP-MIN composite fuzzy relation equations, see [4–7].
Let $R$ be fuzzy relation from $\mathcal{S}$ to $\mathcal{D}$ (i.e. $R \subseteq s^\mathcal{S} \times 2^\mathcal{D}$), with $\mu_R(s, d)$ representing the degree of association of the syndrome $s$ with the clinical characteristics of the diseases in $d$.

The fuzzy relations $R^\#, R^*$ and $R^*_{pr}$ (from $\mathcal{D}$ to $\mathcal{S}$) enable a complete study of inverse problems which consist in the determination of degrees of combination of diseases with groups of symptoms. Such fuzzy relations are characterized by

\begin{align}
\mu_{R^\#}(d, s) &= \sup_{d' \subseteq d, \ d' \neq \emptyset} \mu_R(s, d'), \\
\mu_{R^*}(d, s) &= \sup_{d' \cap d \neq \emptyset} \mu_R(s, d') = \sup_{d' \cap d \neq \emptyset} [\mu_R(s, d') \wedge \mu_{S^\#}(d', d)], \\
\mu_{R^*_{pr}}(d, s) &= \mu_{R^\#}(d, s) \lor \sup_{d' \cap d \neq \emptyset} [\mu_R(s, d') \wedge \mu_S(d', d)] \\
\end{align}

or $R^{*_{pr}} = (S \circ R)^\#$.

Let us note that when $d$ consists of a single diagnosis, e.g. $d = \{\delta\}$, the following property holds

\begin{equation}
\mu_{R^\#}(\delta, s) = \mu_R(s, \delta). \tag{56}
\end{equation}

Moreover, we always have for all $d \subseteq \mathcal{D}$,

\begin{equation}
\forall \delta \in d, \quad \mu_R(s, \delta) \leq \mu_{R^\#}(d, s). \tag{57}
\end{equation}

We suggest to only use $R^\#$ and $R^{*_{pr}}$, but not $R^*$ in which the supremum involved in its definition takes into account some intersections which may not be of paramount significance.

In terms of medical diagnosis assistance we finally indicate a formulation which shows some analogies with already proposed models, see [5, 7, 9].

Let $\mathcal{D}$ be a set of diagnosis, $\mathcal{S}$ a set of symptoms, $\mathcal{G}$ a group of patients. The relation of inclusion between subsets is here replaced by "implications".

For $s \subseteq \mathcal{S}$ and $d \subseteq \mathcal{D}$, $s \rightarrow d$ all patients, with observed symptoms in $s$, must present the diagnosis in $d$.

Moreover, for $s \subseteq \mathcal{S}$ and $d \subseteq \mathcal{D}$, $s \leftrightarrow d \Delta$ all patients, with observed symptoms in $s$, must present the diagnosis in $d$.

Defining $S^\# \subseteq 2^\mathcal{D} \times 2^\mathcal{D}$ and $S^\# \subseteq 2^\mathcal{D} \times 2^\mathcal{D}$ by

\begin{align}
\mu_{S^\#}(s, d) &= 1 \quad \text{for} \quad s \rightarrow d, \\
&= 0 \quad \text{elsewhere}, \\
\mu_{S^\#}(s, d) &= 1 \quad \text{for} \quad s \leftrightarrow d, \\
&= 0 \quad \text{elsewhere}
\end{align}

\begin{align}
\mu_{S^\#}(s, d) &= 1 \quad \text{for} \quad s \rightarrow d, \\
&= 0 \quad \text{elsewhere}, \\
\mu_{S^\#}(s, d) &= 1 \quad \text{for} \quad s \leftrightarrow d, \\
&= 0 \quad \text{elsewhere}
\end{align}
and for $R$ being a fuzzy relation from $\mathcal{P}$ to $\mathcal{S}$, we derive

$$\mu_{R^+}(d, p) = \sup_{s \rightarrow d} \mu_R (p, s) = \sup_s [\mu_R (p, s) \land \mu_{S^+}(s, d)]$$

(60)

$$\mu_{R^*}(d, p) = \sup_{s \rightarrow d} \mu_R (p, s) = \sup_s [\mu_R (p, s) \land \mu_{S^*}(s, d)]$$

(61)

which allows one to infer associations of patients with diagnosis from observed symptoms.

8. Concluding remarks

For further developments of the notions of inverses of fuzzy relations we suggest an exploration of “belief relations”, see [10] for belief functions, for problems encountered in combinations of evidences, where “belief relations” may be associated with possibility distributions.

Finally, in the theory of possibility, the compatibility of possibility measures (in different spaces) with fuzzy relations deserves to be investigated together with particularized possibility distributions which involve SUP–MIN compositions see [13].

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