Two-Reaction Theory of Synchronous Machines
Generalized Method of Analysis—Part I

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Synopsis.—Starting with the basic assumption of no saturation or hysteresis, and with distribution of armature phase m. m. f. effectively sinusoidal as far as regards phenomena dependent upon rotor position, general formulas are developed for current, voltage, power, and torque under steady and transient load conditions. Special detailed formulas are also developed which permit the determination of current and torque on three-phase short circuit, during starting, and when only small deviations from an average operating angle are involved.

In addition, new and more accurate equivalent circuits are developed for synchronous and asynchronous machines operating in parallel, and the domain of validity of such circuits is established. Throughout, the treatment has been generalized to include salient poles and an arbitrary number of rotor circuits. The analysis is thus adapted to machines equipped with field pole collars, or with amortisseur windings of any arbitrary construction.

It is proposed to continue the analysis in a subsequent paper.

This paper presents a generalization and extension of the work of Blondel, Dreyfus, and Doherty and Nickle, and establishes new and general methods of calculating current power and torque in salient and non-salient pole synchronous machines, under both transient and steady load conditions.

Attention is restricted to symmetrical three-phase* machines with field structure symmetrical about the axes of the field winding and interpolar space, but salient poles and an arbitrary number of rotor† circuits is considered.

Idealization is resorted to, to the extent that saturation and hysteresis in every magnetic circuit and eddy currents in the armature iron are neglected, and in the assumption that, as far as concerns effects depending on the position of the rotor, each armature winding may be regarded as, in effect, sinusoidally distributed.‡

A. Fundamental Circuit Equations

Consider the ideal synchronous machine of Fig. 1, and let

\[ i_a, i_b, i_c = \text{per unit instantaneous phase currents} \]
\[ e_a, e_b, e_c = \text{per unit instantaneous phase voltages} \]
\[ \psi_a, \psi_b, \psi_c = \text{per unit instantaneous phase linkages} \]
\[ t = \text{time in electrical radians} \]
\[ p = \frac{d}{dt} \]

Then there is

\[
\begin{align*}
\psi_a &= I_d \cos \theta - I_q \sin \theta \\
\psi_b &= I_d \cos (\theta - 120) - I_q \sin (\theta - 120) \\
\psi_c &= I_d \cos (\theta + 120) + I_q \sin (\theta + 120)
\end{align*}
\]

\[
\begin{align*}
\psi_a &= I_d \cos \theta - I_q \sin \theta \\
\psi_b &= I_d \cos (\theta - 120) - I_q \sin (\theta - 120) \\
\psi_c &= I_d \cos (\theta + 120) + I_q \sin (\theta + 120)
\end{align*}
\]

where,

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†Single-phase machines may be regarded as three-phase machines with one phase open circuited.
‡For numbered references see Bibliography.


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It will be convenient to write \( H (p) = x_d - x_d (p) \) and to rewrite (4) in the form,

\[
I_d = G (p) E + [x_d - x_d (p)] i_d
\]

If there are no additional rotor circuits, there is, as shown in Appendix I,

\[
\Psi = I - (x_d - x_d') i_d
\]

where \( T_o \) is the open circuit time constant of the field in radians.

There is then,

\[
G (p) = \frac{1}{T_o p + 1}
\]

\[
x_d (p) = \frac{x_d' T_o p + x_d}{T_o p + 1}
\]

If there is one additional rotor circuit in the direct axis there is,

\[
\Psi = I + X_{f1d} I_{1id} - (x_d - x_d') i_d = \frac{E - I}{T_o p}
\]

\[
\Psi_{1id} = X_{11d} I_{1id} + X_{f1d} I - x_{m1} d i_d = -\frac{I_{1id}}{T_{01d} p}
\]

which gives,

\[
G (p) = \frac{[X_{11d} - X_{f1d}] T_{01d} p + 1}{A (p)}
\]

\[
T_0 T_{01d} [X_{11d} (x_d - x_d') - X_{f1d} x_{m1d}] p^2
\]

\[
x_d (p) = \frac{x_d - [x_d - x_d'] T_{01d} + x_{m1d} T_o p}{A (p)}
\]

where,

\[
A (p) = [X_{11d} - X_{f1d}] T_{01d} p^2 + [X_{11d} T_0 + T_{01d}] p + 1
\]

If there is more than one additional rotor circuit the operators \( G (p) \) and \( x_d (p) \) will be more complicated but may be found in the same way. The effects of external field resistance may be found by changing the term \( I \) in the field voltage equation to \( RI \). Open circuited field corresponds to \( R \) equal to infinity.

Similarly, there will be

\[
I_4 = [x_d - x_4 (p)] i_4
\]

(5)

where,

\[
i_4 = -\frac{2}{3} \left[ i_a \sin \theta + i_b \sin (\theta - 120) + i_c \sin (\theta + 120) \right]
\]

(3a)

\[
x_4 (p) = x_{4d} \sin \theta + x_{4q} \cos \theta
\]

(7)

So far, 10 equations have been established relating the 15 quantities \( e_a, e_b, e_c, i_a, i_b, i_c, \psi_a, \psi_b, \psi_c, i_d, i_q, I_d, I_q, E, \theta \) in a general way. It follows that when any five of the quantities are known the remaining 10 may be determined. Their determination is very much facilitated, however, by the introduction of certain auxiliary quantities \( e_{d4}, e_{q4}, e_{l4}, i_{d4}, i_{q4}, \psi_{d4}, \psi_{q4}, \psi_{l4} \).

Thus, let

\[
i_0 = \frac{1}{3} [i_a + i_b + i_c]
\]

(3b)

\[
e_{d4} = \frac{2}{3} \left[ e_a \cos \theta + e_b \cos (\theta - 120) + e_c \cos (\theta + 120) \right]
\]

(6)

\[
e_{q4} = -\frac{2}{3} \left[ e_a \sin \theta + e_b \sin (\theta - 120) + e_c \sin (\theta + 120) \right]
\]

\[
e_{l4} = \frac{1}{3} \left[ e_a + e_b + e_c \right]
\]

\[
\psi_{d4} = \frac{2}{3} \left[ \psi_a \cos \theta + \psi_b \cos (\theta - 120) + \psi_c \cos (\theta + 120) \right]
\]

\[
\psi_{q4} = -\frac{2}{3} \left[ \psi_a \sin \theta + \psi_b \sin (\theta - 120) + \psi_c \sin (\theta + 120) \right]
\]

*This definition is somewhat different from that given in reference 2.
\[
\psi_0 = \frac{1}{3} \{ \psi_a + \psi_b + \psi_c \}
\]
then from Equation (1) there is
\[
e_d = \frac{2}{3} \{ \cos \theta \ p \ \psi_a + \cos(\theta - 120) \ p \ \psi_b + \cos(\theta + 120) \ p \ \psi_c \} - r_i_d
\]
\[
e_q = -\frac{2}{3} \{ \sin \theta \ p \ \psi_a + \sin(\theta - 120) \ p \ \psi_b + \sin(\theta + 120) \ p \ \psi_c \} - r_i_q
\]
\[
e_0 = p \ \psi_0 - r \ i_0
\]
but,
\[
p \ \psi_d = \frac{2}{3} \{ \cos \theta \ p \ \psi_a + \cos(\theta - 120) \ p \ \psi_b + \cos(\theta + 120) \ p \ \psi_c \}
\]
\[
- \frac{2}{3} \{ \sin \theta \ \psi_a + \sin(\theta - 120) \ \psi_b + \sin(\theta + 120) \ \psi_c \} \ p \ \theta
\]
\[
= e_d + r \ i_d + \psi_q \ p \ \theta
\]
\[
p \ \psi_q = -\frac{2}{3} \{ \sin \theta \ p \ \psi_a + \sin(\theta - 120) \ p \ \psi_b + \sin(\theta + 120) \ p \ \psi_c \}
\]
\[
- \frac{2}{3} \{ \cos \theta \ \psi_a + \cos(\theta - 120) \ \psi_b + \cos(\theta + 120) \ \psi_c \} \ p \ \theta
\]
\[
= e_q + r \ i_q - \psi_d \ p \ \theta
\]
hence there is
\[
e_d = p \ \psi_d - r \ i_d - \psi_q \ p \ \theta
\]
\[
e_q = p \ \psi_q - r \ i_q + \psi_d \ p \ \theta
\]
\[
e_0 = p \ \psi_0 - r \ i_0
\]
Also it may be readily verified that
\[
\psi_d = I_d - x_d i_d = G \ (p) \ E - x_d \ (p) \ i_d
\]
\[
\psi_q = I_q - x_q i_q = -x_q \ (p) \ i_q
\]
\[
\psi_0 = -x_0 i_0
\]
Equations (8) to (13) establish six relatively simple relations between the 11 quantities \( e_d, e_q, e_0, i_d, i_q, i_0, \psi_a, \psi_b, \psi_c, E, \theta \). In practice it is usually possible to determine five of these quantities directly from the terminal conditions, after which the remaining six may be calculated with relative simplicity. After the direct, quadrature, and zero quantities are known the phase quantities may be determined from the identical relations
\[
i_a = i_d \cos \theta - i_q \sin \theta + i_0
\]
\[
i_b = i_d \cos(\theta - 120) - i_q \sin(\theta - 120) + i_0
\]
\[
i_c = i_d \cos(\theta + 120) - i_q \sin(\theta + 120) + i_0
\]
\[
\psi_a = \psi_d \cos \theta - \psi_q \sin \theta + \psi_0
\]
\[
\psi_b = \psi_d \cos(\theta - 120) - \psi_q \sin(\theta - 120) + \psi_0
\]
\[
\psi_c = \psi_d \cos(\theta + 120) - \psi_q \sin(\theta + 120) + \psi_0
\]
\[
e_a = e_d \cos \theta - e_q \sin \theta + e_0
\]
\[
e_b = e_d \cos(\theta - 120) - e_q \sin(\theta - 120) + e_0
\]
\[
e_c = e_d \cos(\theta + 120) - e_q \sin(\theta + 120) + e_0
\]
Referring to Fig. 2, it may be seen that when there are no zero quantities, that is, when \( e_0 = \psi_0 = i_0 = 0 \), the phase quantities may be regarded as the projection of vectors \( \vec{e}, \vec{\psi}, \vec{i} \) on axes lagging the direct axis by angles \( \theta, \theta - 120 \) and \( \theta + 120 \), where taking the direct axis as the axis of reals,
\[
\bar{\theta} = e_d + j e_q
\]
\[
\bar{\psi} = \psi_d + j \psi_q
\]
\[
\bar{i} = i_d + j i_q
\]
If we introduce in addition the vector quantity,
\[
\bar{I} = I_d + j I_q
\]
the circuit equations previously obtained may be transferred into the corresponding vector forms,
\[
\bar{e} = p \ \bar{\psi} - \bar{n} + [p \ \theta] j \ \bar{\psi}
\]
\[
\bar{\psi} = \bar{I} - \bar{x} \ \bar{n}
\]
where,
\[
\bar{x} \ \bar{n} = x_d \ i_d + j \ x_q \ i_q
\]
Fig. 3 shows these relations graphically.

**B. Armature Power Output**

The per-unit instantaneous power output from the armature is necessarily proportional to the sum
By consideration of any instant during normal operation at unity power factor it may be seen that the factor of proportionality must be 2/3. That is,

\[ P = \text{per-unit instantaneous power output} \]
\[ = \frac{2}{3} \{ e_a i_a + e_b i_b + e_i i_i \} \]

Substituting from Equations (14) and (16) there results the useful relation,

\[ P = e_d i_d + e_q i_q + e_0 i_0 \]  (17)

C. Electrical Torque on Rotor

It is possible to determine the electrical torque on the rotor directly from the general relation,

\[ \{\text{Total power output}\} = \]
\[ \{\text{mechanical power transferred across gap}\} + \{\text{rate of decrease of total stored magnetic energy}\} - \{\text{total ohmic losses}\} \]  (18)

However, since this torque depends uniquely only on the magnitudes of the currents in every circuit of the machine, it follows that a general formula for torque may be derived by considering any special case in which arbitrary conditions are imposed as to the way in which these currents are changing as the rotor moves.

The simplest conditions to impose are that \( I_d, I_q, i_d, i_q, \) and \( i_0 \) remain constant as the rotor moves. In this case there will be no change in the stored magnetic energy of the machine as the rotor moves, and the power output of the rotor will be just equal in magnitude and opposite in sign to the rotor losses. It follows that under the special conditions assumed, Equation (18) becomes simply,

\[ \{\text{armature power output}\} = \]
\[ \{\text{mechanical power across gap}\} - \{\text{armature losses}\} \]

or,

\[ P = T \theta - \frac{2}{3} \{ i_a^2 + i_b^2 + i_i^2 \} \]
\[ = T \theta - r \{ i_d^2 + i_q^2 + i_0^2 \} \]

Then,

\[ T = \text{per-unit instantaneous electrical torque} \]
\[ = \frac{e_d i_d + e_q i_q + e_0 i_0 + r \{ i_d^2 + i_q^2 + i_0^2 \}}{p \theta} \]

but subject to the conditions imposed,

\[ e_d = - \psi_d p \theta - r i_d \]
\[ e_q = \psi_d p \theta - r i_q \]
\[ e_0 = - r i_0 \]

It therefore follows that,

\[ T = i_q \psi_d - i_d \psi_q \]  (19)
\[ = \text{vector product of } \vec{\psi} \text{ and } \vec{i} \]
\[ = \vec{\psi} \times \vec{i} \]  (19a)

a result which could have been established directly by physical reasoning. Formula (19) is employed by Dreyfus in his treatment of self-excited oscillations of synchronous machines.\(^{14}\)

D. Constant Rotor Speed

Suppose that the constant slip of the rotor is \( s \).

Then there is,

\[ e_d = p \psi_d - r i_d - (1 - s) \psi_q \]
\[ e_q = p \psi_q - r i_q + (1 - s) \psi_d \]

but,

\[ \psi_d = G(p) E - x_d(p) i_d \]
\[ \psi_q = - x_q(p) i_q \]

Putting

\[ p x_d(p) + r = z_d(p) \]
\[ p x_q(p) + r = z_q(p) \]

there is

\[ e_d = p G(p) E - z_d(p) i_d + (1 - s) z_q(p) i_q \]  (20)
\[ e_q = (1 - s) [G(p) E - x_d(p) i_d] - z_q(p) i_q \]  (21)

Solving gives,

\[ i_d = \left\{ (p z_q(p) + (1 - s)^2 x_q(p)) G(p) E - z_q(p) e_d - (1 - s) x_q(p) e_q \right\} \div D(p) \]  (22)
\[ i_q = \frac{(1 - s) r G(p) E - z_d(p) e_d + (1 - s) x_d(p) e_d}{D(p)} \]  (23)

where,

\[ D(p) = z_d(p) z_q(p) + (1 - s)^2 x_d(p) x_q(p) \]

E. Two Machines Connected Together

Suppose that two machines which we will designate respectively by the subscripts \( q \) and \( h \), are connected together, but not to any other machines or circuits, and assume in addition that there are no zero quantities. In this case the voltages of each machine will be equal phase for phase, and it therefore follows that the voltage vectors of each machine must coincide, as shown in Fig. 4.

Referring to the figure it will be seen that the direct and quadrature components of voltage of the two machines are subject to the mutual relations,

\[ e_{hd} = e_{hd} \cos \delta - e_{hq} \sin \delta \]
\[ e_{hq} = e_{hd} \sin \delta + e_{hq} \cos \delta \]  (24)
\[ e_{qd} = e_{hd} \cos \delta + e_{hq} \sin \delta \]
\[ e_{qg} = - e_{hd} \sin \delta + e_{hq} \cos \delta \]  (25)
On the other hand, for currents there will be
\[
i_{dq} = -\left( i_{dq} \cos \delta - i_{dq} \sin \delta \right)
\]
\[
i_{hq} = -\left( i_{hq} \cos \delta + i_{hq} \sin \delta \right)
\]
\[
i_{gd} = -\left( -i_{gd} \cos \delta + i_{gd} \sin \delta \right)
\]
\[
i_{gq} = -\left( -i_{gq} \cos \delta - i_{gq} \sin \delta \right)
\]
(26)
(27)

F. One Machine on an Infinite 

In (E), if machine h has zero impedance, it follows from (20) and (21) that \( e_{dh} = 0, e_{hq} = \) bus voltage say = e.

Then for machine g there is,
\[
e_d = e \sin \delta
\]
\[
e_q = e \cos \delta
\]
(28)

G. Torque Angle Relations

From Equations (11), (12), and (19), there is,
\[
T = \frac{I_q \psi_d}{x_q} - \frac{I_q \psi_q}{x_d} - \frac{x_d - x_q}{x_d x_q} \psi_d \psi_q
\]

Then if the rotor leads the vector \( \psi \) by an angle \( \delta \), there is
\[
\psi_q = -\psi \sin \delta
\]
\[
\psi_d = \psi \cos \delta
\]
\[
T = \frac{I_q \psi}{x_q} \cos \delta + \frac{I_q \psi \sin \delta}{x_d} + \frac{x_d - x_q}{2 x_d x_q} \psi \sin 2 \delta
\]
(29)

A derivation of this formula for steady load conditions has been previously given by Doherty and Nickle.9

H. Three-Phase Short Circuit with Constant Rotor Speed

Maintained

Since a three-phase short circuit causes \( e_d \) and \( e_q \) to vanish suddenly, its effect with constant rotor speed maintained may be found by impressing \( e_d = e_{d0} \), \( e_q = -e_{q0} \) in (22) and (23) where \( e_{d0} \) and \( e_{q0} \) are the values of \( e_d \) and \( e_q \) before the short circuit. The initial currents existing before the short circuit must be added to the currents found in this way in order to obtain the resultant current after the short circuit.

With \( s = 0 \) and \( E \) constant there is in detail,
\[
i_d = \frac{z_d(p) e_{d0} + x_q(p) e_{q0}}{D(p)} + \frac{1}{r^2 + x_d x_q} \cdot \frac{x_d E - r e_{d0} - x_q e_{q0}}{D(p)}
\]
\[
i_q = \frac{z_q(p) e_{q0} - x_d(p) e_{d0}}{D(p)} + 1 + \frac{r E - r e_{d0} + x_d e_{q0}}{r^2 + x_d x_q} \cdot \frac{1}{D(p)}
\]
(30)

The working out of the formulas may be illustrated by consideration of the simple case of a machine with no rotor circuits in addition to the field. In this case there is
\[
x_q(p) = x_q
\]
\[
x_d(p) = \frac{x_d' T_0 p + x_d}{T_0 p + 1}
\]
\[
D(p) = \left( \frac{x_d' T_0 p + x_d}{T_0 p + 1} + r \right) (x_q p + r)
\]
Equations (32) by the application of Equations (14).

For the particular case
\[
T_0 = 2,000, x_d = 1.00, x_q = 0.60, x_d' = 0.30
\]
the roots \( \alpha_1, \alpha_5, \alpha_3 \) of the equation \( d(p) = 0 \), were found to be as shown in Figs. 5, 6, and 7, where
\[
\alpha_2 = \alpha_4 + \alpha_6
\]
\[
\alpha_3 = \alpha_4 - \alpha_6
\]
It will be noted that, as would necessarily be the
case, where \( r = 0 \), \( \alpha_1 \) is equal to the reciprocal of the short circuit time constant of the machine, i.e., for \( r = 0 \),

\[
\alpha_1 = - \frac{x_d' - 1}{T_o} = - 0.001667
\]

while for \( r = \infty \)

\[
\alpha_1 = - \frac{1}{T_o} = - 0.000500
\]

The root \( \alpha_a \) is found to be almost exactly equal to the value which it would have were \( T_o = \infty \), i.e.,

\[
\alpha_a = \frac{r (x_d' + x_q)}{2 x_d' x_q} \text{ approximately}
\]

Thus, in the special case considered this approximate formula gives

\[
\alpha_a = \frac{(0.30 + 0.60) r}{2 \times 0.30 \times 0.60} = 2.50 r
\]

which checks the result found by the exact solution of the cubic.

I. **Starting Torque**

On infinite bus and with slip \( s \), there will be, choosing

\[
e_d = \cos st
\]

\[
e_q = \sin st
\]

If we now introduce a system of vectors rotating at \( s \) per-unit angular velocity there is

\[
ed = 1.0
\]

\[
e_q = - j
\]

\[
p = j s
\]

Then from (22) and (23),

\[
i_d = \{ j s x_d (j s) + r - j (1 - s) x_d (j s) \} + \{ j s x_d (j s) + r [j s x_d (j s) + r] + (1 - s)^2 x_d (j s) x_q (j s) \}
\]

\[
= \frac{j (1 - 2 s) x_d (j s) x_q (j s) - r}{r^2 + (1 - 2 s) x_d (j s) x_q (j s) + j s r [x_d' (j s) + x_q' (j s)]}
\]

\[
= \left\{ \begin{array}{l} j x_d (j s) - \frac{r}{1 - 2 s} + \{ x_d (j s) x_q (j s) \\ + \frac{r}{1 - 2 s} [r + j s (x_d (j s) + x_q (j s))] \end{array} \right. \}
\]

\[
i_q = - \frac{[j s x_d (j s) + r] (- j) - (1 - s) x_d (j s)}{r^2 + (1 - 2 s) x_d (j s) x_q (j s) + j s r [x_d' (j s) + x_q' (j s)]}
\]

\[
= \left\{ \begin{array}{l} x_d (j s) + \frac{j r}{1 - 2 s} + \{ x_d (j s) x_q (j s) \\ + \frac{r}{1 - 2 s} [r + j s (x_d (j s) + x_q (j s))] \end{array} \right. \}
\]

The expressions for average power and torque then become,

\[
P_{av} = 1/2 [e_d \cdot i_d + e_q \cdot i_q]
\]

\[
T_{av} = 1/2 [e_q \cdot \psi_d - i_d \cdot \psi_q]
\]

where the dot indicates the scalar product, or

\[
P_{av} = 1/2 [e_d \cdot i_d - j \cdot i_q]
\]

\[
= 1/2 \text{[Real of } i_q - \text{ Imaginary of } i_q]\]

There is in general,

\[
\psi_d = \frac{e_d + r i_d - (1 - s)}{e_q + r i_q}
\]

\[
= \frac{p (e_d + r i_d) + (1 - s) (e_q + r i_q)}{p^2 + (1 - s)^2}
\]

\[
\psi_q = \frac{p (e_q + r i_q) - (1 - s) (e_d + r i_d)}{p^2 + (1 - s)^2}
\]
\[ \psi_d = \frac{js(e_d + r i_d) + (1 - s)(e_q + r i_q)}{1 - 2s} \]

\[ \psi_q = \frac{js(e_q + r i_q) - (1 - s)(e_d + r i_d)}{1 - 2s} \]

with \( e_d = 1.0, e_q = -j \)

\[ \psi_d = \frac{j s + j s r i_d + (1 - s)(-j) + (1 - s)r i_d}{1 - 2s} \]

\[ = \frac{-(1 - 2s)j + r[j s i_d + (1 - s)i_q]}{1 - 2s} \]

\[ = -j + \frac{r}{1 - 2s} [j s i_d + (1 - s)i_q] \quad (39) \]

\[ \psi_q = \frac{j s (-j + r i_q) - (1 - s) - r(1 - s) i_d}{1 - 2s} \]

\[ = \frac{- (1 - 2s) + r[j s i_q - (1 - s)i_d]}{1 - 2s} \]

\[ = -1 + \frac{r}{1 - 2s} [j s i_q - (1 - s)i_d] \quad (40) \]

Thus,

\[ T_{av} = \frac{1}{2} \begin{bmatrix} i_q \cdot (-j) + i_q \cdot \frac{r}{1 - 2s} (j s i_d + (1 - s)i_q) \\ -i_d \cdot (-1) - i_d \cdot \frac{r}{1 - 2s} (j s i_q - (1 - s)i_d) \end{bmatrix} \]

\[ = P_{av} + \frac{r}{2(1-2s)} \left[ (1 - s)(i_q^2 + i_d^2) \right] + 2s \cdot j \cdot i_d \]

\[ = P_{av} + \frac{r}{2} (i_q^2 + i_d^2) + \frac{r}{2(1-2s)} (i_q + j i_d)^2 \quad (41) \]

Mr. Ralph Hammar, who has been engaged in the application of the general method of calculation outlined above, to the predetermination of the starting torque of practical synchronous motors, has suggested an interesting modification of formulas (36) and (41), based upon the fact that, since the total m. m. f. consists of direct and quadrature components pulsating at slip frequency, it may be resolved into two components, one moving forward at a per-unit speed \( 1 - s + s = 1.0 \), and the other moving backward at a per-unit speed \( 1 - s - s = 1 - 2s \). Thus from this standpoint half of both the direct and quadrature components will move forward, and half backward. Since the quadrature axis is ahead of the direct it follows that as far as concerns the forward component the quadrature current \( i_q \) is equivalent to a d-c. \( j i_q \), while as regards backward component it is equivalent to a direct component \( -i_q \). It follows that the vector amounts of forward and backward m. m. f. or current are

forward current \( i_f = \frac{1}{2} (i_d + j i_q) \)

backward current \( i_b = \frac{1}{2} (i_d - j i_q) \quad (42) \)

If we define by analogy,

forward voltage \( = \frac{1}{2} (e_d + j e_q) \)

backward voltage \( = \frac{1}{2} (e_d - j e_q) \quad (43) \)

There is,

\[ i_f = \frac{1}{2} \left[ \frac{-2r}{1 - 2s} + j[x_d(js) + x_q(js)] \right] + x_d(js) + r [x_d(js) + x_q(js)] \]

\[ + x_q(js)] \quad (44) \]

\[ i_b = \frac{1}{2} \left[ j[x_q(js) - x_d(js)] \right] + x_d(js) + r [x_d(js) + x_q(js)] \]

\[ + x_q(js)] \quad (45) \]

\[ P_{av} = e_f \cdot i_f \] real of \( i_f \quad (46) \]

\[ T_{av} = P_{av} + r i_f^2 + \frac{r}{1 - 2s} i_q^2 \quad (47) \]

J. Zero Armature Resistance, One Machine Connected to an Infinite Bus

Assume that a machine of negligible armature resistance is operating from an infinite bus of per-unit voltage \( e \), at synchronous speed, with a steady excitation voltage \( E_0 \) and displacement angle \( \delta \). At the instant \( t = 0 \), let \( \delta \) and \( E \) change.

There is,

\[ i_d = \frac{E_0 - \psi_d}{x_d} - \frac{1}{x_d(p)} \Delta \psi_d + \frac{G(p)}{x_d(p)} \Delta E \]

\[ i_q = -\frac{\psi_q}{x_q} - \frac{1}{x_q(p)} \Delta \psi_q \]

\[ \psi_d = e \cos \delta \]

\[ \psi_q = -e \sin \delta \]

From which there is, by obvious re-arrangement,
Then,

\[ T = \frac{E}{x_d} \sin \delta + \frac{x_d - x_q}{2x_d x_q} e^\delta \sin 2\delta \]

\[ + e^\delta \cos \delta \frac{x_d - x_q}{x_d x_q} (\sin \delta - \sin \delta_0) \]

\[ + e^\delta \sin \delta \frac{x_d - x_d}{x_d x_d} (\cos \delta_0 - \cos \delta) \]

\[ - \sin \delta \frac{x_d (p) - x_d G (p)}{x_d x_d} \cdot \Delta E \]

But quantities \( a_{dn}, a_{n}, \alpha_{dn}, \alpha_{n}, b_n, \beta_n \) may be found such that

\[ \frac{x_d - x_q}{x_d (p)} \cdot 1 = \frac{x_d - x_q}{x_q} \sum a_{dn} e^{-\alpha_{dn} u} \]

\[ + e^{\delta} \cos \frac{x_d - x_q}{x_d x_q} \sum a_{n} e^{-\alpha_{n} u} \]

\[ + e^{\delta} \sin \frac{x_d - x_d}{x_d x_d} \sum b_n e^{-\beta_{n} u} \]

\[ x_q = x_q (\infty) \]

\[ x_d = x_d (\infty) \]

\[ \sum a_{dn} = 1.0 \]

\[ \sum a_{n} = 1.0 \]

\[ \sum b_n = 1.0 \]

It therefore follows from the operational rule that,

\[ f (p) F (t) = F (p) \phi (t) + \int_0^t \phi (t - u) F' (u) d u \]

where,

\[ \phi (t) = f (p) \cdot 1 \]

that if

\[ \delta = \delta (t) \]

\[ p \delta = \delta' (t) \]

\[ \Delta E = \Delta E' (t) \]

\[ p \Delta E = \Delta E' (t) \]

Equations (48) and (49) may be rewritten in the form,

\[ \frac{E - e \cos \delta }{x_d} \]

\[ + e^{\delta} \cos \frac{x_d - x_q}{x_d x_q} \sum a_{dn} e^{-\alpha_{dn} u} \int_0^t e^{\alpha_{dn} u} \sin \delta (u) \delta' (u) d u \]

\[ - \frac{1}{x_d} \sum b_n e^{-\beta_{n} u} \int_0^t e^{\beta_{n} u} \Delta E' (u) d u \]  

(48a)

\[ i_q = \frac{-e \sin \delta }{x_q} \]

\[ + e^{\delta} \cos \frac{x_d - x_q}{x_d x_q} \sum a_{n} e^{-\alpha_{n} u} \int_0^t e^{\alpha_{n} u} \cos \delta (u) \delta' (u) d u \]

\[ T = \frac{E}{x_d} \sin \delta + \frac{x_d - x_q}{2x_d x_q} e^{\delta} \sin 2\delta \]

\[ + e^{\delta} \cos \delta \frac{x_d - x_q}{x_d x_q} (\sin \delta - \sin \delta_0) \]

\[ + e^{\delta} \sin \delta \frac{x_d - x_d}{x_d x_d} (\cos \delta_0 - \cos \delta) \]

\[ - \sin \delta \frac{x_d (p) - x_d G (p)}{x_d x_d} \cdot \Delta E \]

Formul (49a) may be used to determine starting torque and current with zero armature resistance, by introducing \( \delta (t) = t, \delta' (t) = s \). Thus the average component of torque is found to be,

\[ T_{av} = \frac{1}{2} \frac{x_d - x_d}{x_d x_d} \sum a_{dn} \frac{\alpha_{dn} s}{\alpha_{dn}^2 + s^2} \]

\[ + \frac{1}{2} \frac{x_d - x_d}{x_d x_d} \sum a_{n} \frac{\alpha_{n} s}{\alpha_{n}^2 + s^2} \]

(52)

Since

\[ \frac{\alpha s}{\alpha^2 + s^2} \]

is never greater than \( \frac{1}{2} \), and

\[ \sum a_{dn} = \sum a_{n} = 1.0 \]

it follows that \( T_{av} \) is never greater than

\[ \frac{1}{4} \left( \frac{x_d - x_d}{x_d x_d} + \frac{x_d - x_d}{x_d x_d} \right) \]

Equation (53) thus provides a very simple criterion of the maximum possible starting torque of a synchronous motor of given dimensions, when armature resistance is neglected.

The same formula may also be used to obtain a simple expression for the damping and synchronizing components of pulsating torque due to a given small angular pulsation of the rotor.

Thus if the angular pulsation is

\[ \Delta \delta = |\Delta \delta| \sin (s t) \]

and if the pulsation of torque is expressed in the form
\[ \Delta T = T_e \Delta \delta + T_d \frac{d}{dt} \Delta \delta \]

there results,

\[ T_e = T_{e0} + e^2 \sin^2 \delta_0 \frac{x_d - x_d''}{x_d x_d''} \sum \frac{a_{an} s^2}{(\alpha_{an})^2 + s^2} \]

\[ + \frac{e^2 \cos^2 \delta_0}{x_q x_q''} \sum \frac{a_{cn} s^2}{(\alpha_{cn})^2 + s^2} \]

\[ = e^2 \sin^2 \delta_0 \frac{x_d - x_d''}{x_d x_d''} \sum \frac{a_{an} s \alpha_{an}}{(\alpha_{an})^2 + s^2} \]

\[ + e^2 \cos^2 \delta_0 \frac{x_q - x_q''}{x_q x_q''} \sum \frac{a_{cn} s \alpha_{cn}}{(\alpha_{cn})^2 + s^2} \]

where,

\[ T_{e0} = \frac{e I_{d0} \cos \delta_0}{x_d} + \frac{e^2 (x_d - x_d)}{x_d x_q} \cos 2 \delta_0 \]

\[ \delta_0 = \text{average angular displacement}, \quad i.e., \quad \text{total angle} = \delta = \delta_0 + \Delta \delta. \]

It can be shown that for the case of no additional rotor circuits, Equations (54) are exactly equivalent to Equations (24) and (25) in Doherty and Nickle’s paper, Synchronous Machines III. The new formulas herein developed are, however, very much simpler in form, especially since in the case which Doherty and Nickle have treated, there is only one term in the summation; that is, \( n = 1 \), and \( \alpha \) is merely the reciprocal of the short circuit time constant of the machine, expressed in radians.

K. The Equivalent Circuit of Synchronous Machines Operating in Parallel at No Load, Neglecting the Effect of Armature Resistance

Let,

\[ \delta_a = \text{angle of rotor } a \text{ and bus} \]

\[ \theta_b = \text{angle of rotor } a \text{ in space} \]

In general, the shaft torque of a machine depends on its acceleration and speed in space, and the magnitude and rate of change of the bus voltage as a vector. If all of the machines are operating at no load and if there is no armature resistance, a small displacement of any one machine will change the magnitude of the bus voltage only by a second order quantity; consequently for small displacements the magnitude of the bus voltage may be regarded as fixed, and only the angle of the bus and rotor need be considered. Furthermore, the electrical torque may be found in terms of \( \delta \) by employing an infinite bus formula. But Equation (49a) implies the alternative general operational form,

\[ T = \frac{e I_{d0} \sin \delta}{x_d} + \frac{e^2 (x_d - x_d)}{2 x_d x_q} \sin 2 \delta \]

\[ - \frac{x_d - x_d''}{x_d x_d''} \frac{e^2 \sin \delta}{x_q x_q''} \sum \frac{a_{an} p}{p + \alpha_{an}} \cos \delta \]

\[ + \frac{x_q - x_q''}{x_q x_q''} e^2 \cos \delta \sum \frac{a_{ae} p}{p + \alpha_{ae}} \sin \delta \]

Therefore in the case under consideration there is for machine \( a \),

\[ T_a = \left[ \frac{e I_a}{x_{da}} + e^2 \frac{(x_{da} - x_{qa})}{x_{da} x_{qa}} \right] \delta_a \]

\[ + \frac{x_{qa} - x_{qa''}}{x_{qa} x_{qa''}} e^2 \sum \frac{a_{ae} p}{p + \alpha_{ae}} \delta_a \]

where: \( e = \text{per-unit bus voltage} \]

\( I_a = \text{per-unit excitation of machine } a \), etc.

This equation can be represented by Fig. 8, in which the charge through the circuit represents the electrical torque of the machine \( (T_a) \).

The capacitances and resistances must be chosen so that

\[ C_{0a} = \frac{x_{da} x_{qa}}{e I_a x_{qa} + e^2 (x_{da} - x_{qa})} \]

\[ C_{na} = \frac{x_{qa} x_{qa''}}{e^2 a_{ae} (x_{qa} - x_{qa''})} \]

\[ R_{oa} = \frac{1}{C_{na} \alpha_{ae}} \]

The equation for the mechanical torque is

\[ T_{sa} = T_a + M_a p s_a \]

where:

\[ M_a = \text{inertia factor of machine } a \text{ in radians} \]

\[ 2 \times \text{stored mech. energy at normal speed} \]

\[ \text{base power} \]

\[ 0.462 \ W R^2 \left( \frac{\text{rev. per min.}}{1000} \right)^2 \]

\[ = 2 \pi I \ \text{base kw.} \]

\( s_a = \text{per-unit speed of machine } a \)

\[ t = \text{time in seconds} \left( \frac{p = \frac{d}{dt}}{p \theta_a} \right) \]

But,

\[ s_a = p \theta_a \]

Thus there is

\[ T_{sa} = T_a + M_a p^2 \theta_a \]

which corresponds to the equivalent circuit of Fig. 9, in which change \( = \theta_a \)

\[ L_a = M_a \]

The machine operating on an infinite bus can be
represented by the equivalent circuit of Fig. 10, since the condition
\[ \theta_a = \delta_a = 0 \]
is fulfilled.

Several machines in parallel on the same bus may be represented by the diagram of Fig. 11, since the conditions
\[ \theta_a - \delta_a = \theta_b - \delta_b = \ldots (= \text{bus angle in space}) \]
\[ T_a + T_b + T_c, \text{ etc.} = \text{bus power output} = 0 \]
A transmission line may be represented by a condenser.
Thus two machines connected by a line of reactance \( x \) would be represented by the circuit of Fig. 12, where
\[ C = \frac{x}{e^2} \quad (58) \]

Shaft torques are, of course, represented by voltages.

Mechanical damping, such as that due to a fan on a motor shaft or that due to the prime mover, is represented by resistance in series with the inductance \( L \) as in Fig. 13. \( R \) must be chosen equal to the rate of decrease in available driving torque with increase in speed.

Governors and other prime mover characteristics may also be represented by connecting their circuits in the inductive branch of the circuit. Thus a governor which acts through a single time constant may be represented by the circuit of Fig. 14, where

\[ R_g = \frac{1}{\text{regulation}} \]
\[ C_g = \frac{\text{time constant of governor in elec. radians}}{R_g} \quad (59) \]
An induction motor is represented by the simple circuit of Fig. 15 and is precisely the circuit of a synchronous machine with only one time constant and \( C_0 = \infty \) on account of \( I = 0 \).

Results similar to these have been previously shown by Arnold, Nickle, and others, but simpler and more approximate circuits were used, the branches of the several circuits were not directly evaluated in terms of machine constants, and the derivation was incomplete in that the limitation to no load and zero resistance was not appreciated.

L. Torque Angle Relations of a Synchronous Machine Connected to an Infinite Bus, for Small Angular Deviations from an Average Operating Angle

There is, in general,

\[
T = T_0 + \Delta T = (\psi_{d0} + \Delta \psi_d) (i_{q0} + \Delta i_q) - (i_{d0} + \Delta i_d) (\psi_{q0} + \Delta \psi_q)
\]

For small angular deviations,

\[
\Delta T = i_{q0} \Delta \psi_d + \psi_{d0} \Delta i_q - i_{d0} \Delta \psi_q - \psi_{q0} \Delta i_d
\]

\[= [\psi_{d0} + i_{d0} x_d(p)] \Delta i_q - [\psi_{q0} + i_{q0} x_q(p)] \Delta i_d \quad (60)
\]

\[
e_{q0} + \Delta e_q = p \Delta \psi_q - r(i_{d0} + \Delta i_d) - (\psi_{d0} + \Delta \psi_d)(1 + p \Delta \delta)
\]

\[
e_{d0} + \Delta e_d = p \Delta \psi_d - r(i_{q0} + \Delta i_q) + (\psi_{d0} + \Delta \psi_d)(1 + p \Delta \delta)
\]

\[
\Delta e_d = p \Delta \psi_d - r \Delta i_d - \psi_{q0} p \Delta \delta - \Delta \psi_d
\]

\[
\Delta e_q = p \Delta \psi_q - r \Delta i_q + \psi_{d0} p \Delta \delta + \Delta \psi_d
\]

from which there is

\[
z_d(p) \Delta i_d - x_q(p) \Delta i_q = - \Delta e_d - \psi_{q0} p \Delta \delta
\]

\[
z_q(p) \Delta i_q + x_d(p) \Delta i_d = - \Delta e_q + \psi_{d0} p \Delta \delta
\]

\[
\Delta i_d = \frac{z_q(p)(- \Delta e_d - \psi_{q0} p \Delta \delta) + x_q(p)(- \Delta e_q + \psi_{d0} p \Delta \delta)}{D(p)}
\]

\[
\Delta i_q = \frac{z_d(p)(- \Delta e_d - \psi_{q0} p \Delta \delta) - x_d(p)(- \Delta e_q - \psi_{d0} p \Delta \delta)}{D(p)}
\]

where,

\[
D(p) = z_d(p) z_q(p) + x_d(p) x_q(p)
\]

but from Equations (28),

\[
e_{d0} + \Delta e_d = e \sin (\delta_0 + \Delta \delta)
\]

\[
e_{q0} + \Delta e_q = e \cos (\delta_0 + \Delta \delta)
\]

\[
\Delta e_d = e \cos \delta_0 \Delta \delta
\]

\[
\Delta e_q = - e \sin \delta_0 \Delta \delta
\]

(62)

\[
\Delta i_d = \frac{-(e \cos \delta_0 + \psi_{q0} p) z_q(p) + (e \sin \delta_0 + \psi_{d0} p) x_d(p)}{D(p)} \Delta \delta
\]

(63)

\[
\Delta i_q = \frac{(e \sin \delta_0 + \psi_{q0} p) z_d(p) + (e \cos \delta_0 + \psi_{d0} p) x_q(p)}{D(p)} \Delta \delta
\]

(64)

say,

\[
\Delta T = f(p) \cdot \Delta \delta
\]

From (57a) the equation for shaft torque becomes

\[
\Delta T_s = (M p^2 + f(p)) \cdot \Delta \delta
\]

Thus,

\[
\Delta \delta = \frac{1}{M p^2 + f(p)} \cdot \Delta T_s
\]

(65)

Appendix

Formula for Linkages and Voltage in Field Circuit with no Additional Rotor Circuits

In this case the per-unit field linkages will depend linearly on the armature and field currents. That is, in general,

\[
\psi = a I - b i_d
\]

Then if normal linkages are defined as those existing at no load there must be \( a = 1.0 \).

The quantity \( b \) may be found by suddenly impressing terminal linkages \( \psi_d \) with no initial currents in the machines and \( E = 0 \).

By definition there is, initially

\[
i_d = - \frac{\psi_d}{x_d'}
\]
but also there must be from the definition of $x_d$:

$$i_d = \frac{I - \psi_d}{x_d}$$

hence there must be an initial induced field current of amount

$$I = \psi_d \left(1 - \frac{x_d}{x_d'} \right)$$

But, initially the field linkages are zero, thus

$$\Psi = \psi_d \left(1 - \left[\frac{x_d}{x_d'} + \frac{b}{x_d'}\right]\right) = 0$$

hence

$$b = x_d - x_d'$$

Similarly, there will be

$$E = \text{per-unit field voltage} = c \cdot \psi + d \cdot I$$

Normal field voltage will be here defined as those existing at no load and normal voltage. This requires that $d = 1$. The quantity $c$ may then be recognized as the time constant of the field in radians when the armature is open-circuited, since with the field shorted under these conditions there is

$$(T_0 p + 1) I = 0$$

$$c \cdot \psi + I = 0$$

$$\psi = I$$

$$c = T_0 = \text{time constant of field with armature open-circuited.}$$

**Bibliography**


**Discussion**

H. C. Specht: I should think Mr. Park's theory could be applied just as well to the so-called synchronous induction motor, that is an induction motor in which the rotor teeth between the poles are cut out for a distance of about one-third of the pole pitch. Such a motor runs at synchronous speed. However, the pull-out torque is much less than that of an induction motor with the full number of teeth.

C. MacMillan: There was one statement in the first page of Mr. Park's paper to the effect that "Idealization is resorted to, to the extent that saturation and hysteresis in every magnetic circuit and eddy currents in the armature iron are neglected...". And with regard to Fig. 5, Mr. Park remarked that it represented a rigorous solution. Perhaps Mr. Park could give us a little more insight into the effect of taking into account saturation, and give other cases in which certain elements have been neglected with more or less effect upon the final results.

W. J. Lyon: In a paper of this description, certain premises should be chosen and, with these always in mind, the mathematical development should be rigorous. The paper may then be criticized either because of insufficient premises or because of incorrect mathematical development. I believe that the former is the kinder method; it is the one I shall employ.

The premises that Mr. Park chooses are that the field and armature windings are symmetrical, that saturation and hysteresis are neglected and that the armature windings are in effect sinusoidally distributed. I take this last to mean that the air-gap flux due to the armature currents is sinusoidally distributed, for if the armature windings themselves were sinusoidally distributed, there would be produced space harmonics in the air-gap flux distribution due to the saliency of the poles, which, as we all know, would complicate the problem tremendously. In order that the mathematical method used by Mr. Park shall be rigorous, I believe it is necessary to make one further assumption. I think I can best explain this by asking you to consider the result of supplying the field winding with a sinusoidal current while the armature rotates at some speed which may be called synchronous. Under these conditions, there will first be produced in the armature windings two sets of balanced currents each of which will produce 3 component flux distributions in the gap. The first of these is what would be produced if the air-gap were uniform, and is proportional to 1/2 (x_d + x_q - x_d'), where x_d equals the armature leakage reactance. The second of these components is proportional to 1/2 (x_d - x_q). The third component is of the same size as the second. Using the values that Mr. Park gives under Section H of his paper, the relative magnitudes of these components would be 0.8 and 0.2. The first and second components react on the field, and produce in it a current of the impressed field frequency. These are the components that Mr. Park has recognized, but the third component produces an entirely different frequency in the field, which will then be reflected into the armature and the process will be repeated. That is, in this respect, it is similar to the condition that exists in a single-phase alternator. As far as I am aware, the Heaviside operational method cannot be used to obtain a rigorous solution for the single-phase alternator. In spite of this, I think the objection that I have raised is of no more importance than the effect of neglecting saturation or...